Olympiad Tricks for Capacitors and Inductors

August 23, 2025

1 Background

Last week, we introduced two new exciting circuit components, *circuits* and *inductors*, and learned their V-I relationships. We then considered how the charge on a capacitor and current in an inductor changes over time in simple circuits, and found that they have characteristic time scales of $\tau = RC$ and $\tau = L/R$, respectively. Today, we will move to learning olympiad tricks for solving actual olympiad problems with capacitors and inductors.

The structure of today's handout is that we will first derive a useful result: the energy stored in an inductor and capacitor. Instead of me walking you through the derivation here in the introduction, I have framed this as a typical olympiad problem, hwere you are walked through the derivation step-by-step. Enjoy! (and make not of the results, they will be important to remember).

Then, for much of *Trying out the waters* you will learn to internalize and use the following insight (text directly from Jaan Kalda's excellent IPhO handouts):

> For circuits containing L and R or C and R
> **At time-scales much shorter than the characteristic times**
>
> $$\tau = RC \quad \text{or} \quad \tau = \frac{L}{R},$$
>
> the capacitor's charge and inductor's current remain almost constant. In particular, if a capacitor was chargeless, its voltage remains almost zero, i.e. it is essentially short-circuited; if there was no current in an inductor, its current remains zero, i.e. the wire leading to the inductor can be considered as broken.
>
> If a capacitor had a charge Q corresponding to a voltage V_0, its voltage remains essentially constant, i.e. it acts as (and can be substituted by) a battery of emf $\mathcal{E} = V_0$. Similarly, if an inductor had a current I_0, it can be substituted by a respective constant current source.
>
> **At time-scales which are much longer than the characteristic times**, the situation is reversed: the inductor can be considered as a short-circuiting wire, and the capacitor as an insulator. This is because all the currents and voltages tend exponentially towards the equilibrium state, so that the difference from the equilibrium value Δ behaves as
>
> $$\Delta \propto e^{-t/\tau}.$$
>
> Thus, the capacitor charge is almost constant, hence there is no current, and the inductor current is almost constant, hence no electromotive force.

Lastly, in *Exploring the deep*, we will learn to make considerations for equilibria states of capacitors. You should think hard (and discuss with your friends) what the conditions are on the charge and voltage of the capacitors in equilibrium.

The final quesiton is a follow-up from last week, for the curious!

2 Questions

2.1 Derive the basics

1. **Deriving the Energy Stored in a Capacitor**

   You know that for a capacitor, the relationship between charge Q, voltage V, and capacitance C is:

   $$Q = CV.$$

   Starting from an **uncharged** capacitor, you slowly (quasi-statically) add charge until it has a final charge Q and final voltage V. At any moment during the charging, let q represent the charge already on the capacitor.

   Your goal is to **derive an expression for the energy U stored in a capacitor** by completing the following steps:

   (a) Show that the instantaneous voltage $v(q)$ across the capacitor when it holds a charge q can be expressed in terms of q and C.

   (b) A small amount of work dW is required to move a tiny charge dq onto the capacitor when the voltage is v. Using "work = charge × voltage," find an expression for dW in terms of v and dq.

   (c) Substitute your result from part (a) into the expression from part (b) so that dW is written *only* in terms of q, dq, and C.

   (d) Integrate dW from $q = 0$ (uncharged) to $q = Q$ (final charge) to find the total energy stored in the capacitor, U. Your final answer for U should be expressed in terms of Q and C.

   (e) Use the relationship $V = Q/C$ to rewrite your expression for U in terms of Q and V.

   (f) Substitute $Q = CV$ into your expression from part (e) to obtain the final formula for the stored energy, U, in terms of C and V.

   (g) Verify that your final expression has units of energy (joules) by showing that:

   $$[\text{farad}] \cdot [\text{volt}]^2 = \text{joule}.$$

   (h) **Quick practice:** A capacitor of $100\,\mu$F is charged to 9 V. Use your final formula to compute the stored energy U in joules.

2. **Deriving the Energy Stored in an Inductor**

   You know that for an inductor, the relationship between the voltage across it, V, the current through it, I, and its inductance, L, is:

   $$V = L\frac{dI}{dt}.$$

   Starting from zero current, you slowly (quasi-statically) increase the current in the inductor until it reaches a final value I. The goal is to **derive an expression for the energy U stored in the inductor** by completing the following steps:

   (a) Starting from $V = L\frac{dI}{dt}$, express the instantaneous voltage v in terms of L and $\frac{dI}{dt}$.

   (b) A tiny amount of work dW is required to push a small amount of charge dq through the inductor when the voltage is v. Using "power = voltage × current," express the power delivered to the inductor as:

   $$P = v \cdot I.$$

   Then, write the infinitesimal work dW in terms of v, I, and dt.

   (c) Substitute $v = L\frac{dI}{dt}$ into your expression for dW so that dW is written only in terms of L, I, and dI.

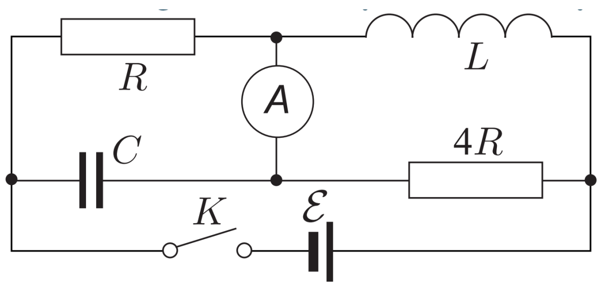

Figure 1:

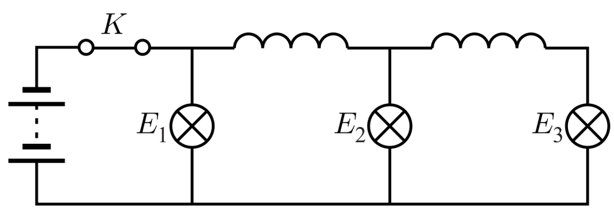

Figure 2:

(d) Integrate your expression for dW from $I = 0$ to $I = I$ to find the total energy stored in the inductor, U. Your answer for U should be expressed in terms of L and I.

(e) Verify that your final expression has units of energy (joules) by showing that:

$$[\text{henry}] \cdot [\text{ampere}]^2 = \text{joule}.$$

(f) **Quick practice:** An inductor with $L = 2.0\,\text{H}$ carries a current of $3.0\,\text{A}$. Use your final formula to compute the stored energy U in joules.

2.2 Try out the waters

3. A capacitor of capacitance C is charged using a battery of electromotive force E. Find the heat dissipated during the charging process (either via a spark or in the wires or in the battery due to (internal) resistance (Kalda).

4. The key of the circuit in figure 1 given below has been kept open; at certain moment, it is closed. (Kalda)

   (a) What is the ammeter reading immediately after the key is closed?

   (b) The key is kept closed until an equilibrium state is achieved; what is the ammeter reading now?

   (c) Now, the key is opened, again; what is the ammeter reading immediately after the key is opened? (Kalda)

5. There are three identical lamps which are connected to a battery as shown in figure 2; the current through each lamp is I. Find the currents immediately after the key is opened.

2.3 Exploring the deep

6. Consider tehe circuit in figure 3. The switch is first in position A until the system has obtained a stationary state. Then the switch is changed to position B. Find the charges on the three capacitors after a long time when a new stationary state is established reached. (NPhO 2022)

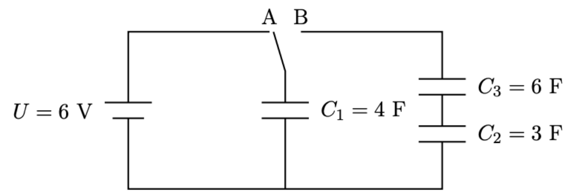

Figure 3: Capacitors with switch

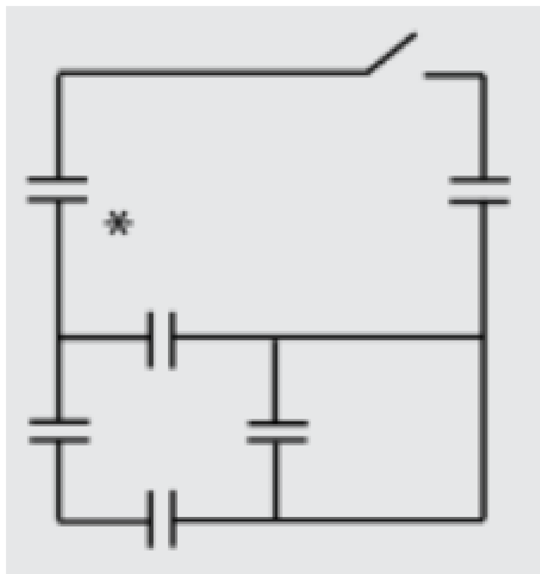

Figure 4:

7. Consider the circuit in figure 4. In this circuit, all capacitors are equal. Initially, the switch is open, and only the capacitor marked with * is charged. The switch is closed, and after the charges have reached a new equilibrium, the capacitor marked with * has acquired a charge Q. What was the initial charge Q_0 on this capacitor? (NPhO 2021)

8. Determine the time constant for the circuit shown in figure 5 (i.e. for the process of charging the capacitor, time interval during which the charging rate drops by a factor of $1/e$). (Kalda)

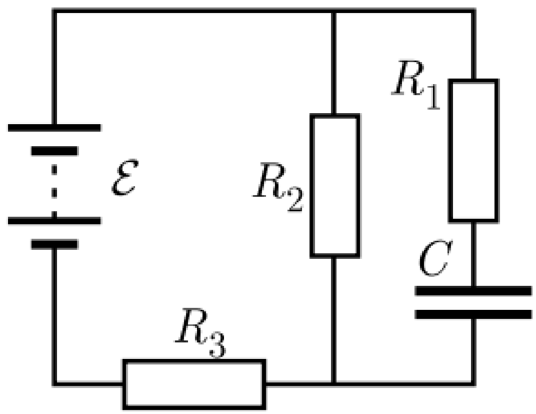

Figure 5:

3 Answers

1. **Energy Stored in a Capacitor**

   (a) The instantaneous voltage across the capacitor is:

   $$v(q) = \frac{q}{C}.$$

   (b) A small amount of work dW needed to add a tiny charge dq is:

   $$dW = v\,dq.$$

   (c) Substituting $v = \frac{q}{C}$ gives:

   $$dW = \frac{q}{C}\,dq.$$

   (d) Integrate from $q = 0$ to $q = Q$:

   $$U = \int_0^Q \frac{q}{C}\,dq = \frac{1}{C} \cdot \frac{q^2}{2}\bigg|_0^Q = \frac{Q^2}{2C}.$$

   (e) Using $V = \frac{Q}{C}$, we can write:

   $$U = \frac{1}{2}QV.$$

   (f) Substituting $Q = CV$ gives the standard form:

   $$U = \frac{1}{2}CV^2.$$

   (g) **Units check:**

   $$[C] = \mathrm{F}, \quad [V] = \mathrm{V},$$

   so:

   $$[C][V]^2 = \mathrm{F} \cdot \mathrm{V}^2 = \frac{\mathrm{C}}{\mathrm{V}} \cdot \mathrm{V}^2 = \mathrm{C} \cdot \mathrm{V} = \mathrm{J}.$$

   (h) **Quick practice example:** For $C = 100\,\mu\mathrm{F}$ and $V = 9\,\mathrm{V}$:

   $$U = \frac{1}{2} \cdot (100 \times 10^{-6}\,\mathrm{F}) \cdot 9^2 = 4.05 \times 10^{-3}\,\mathrm{J}.$$

2. **Energy Stored in an Inductor**

   (a) The voltage across an inductor is given by:

   $$v = L\frac{dI}{dt}.$$

   (b) The instantaneous power delivered to the inductor is:

   $$P = v \cdot I.$$

   Since power is the rate of work, we have:

   $$dW = P\,dt = v \cdot I\,dt.$$

(c) Substituting $v = L\frac{dI}{dt}$:

$$dW = L\frac{dI}{dt} \cdot I\,dt = LI\,dI.$$

(d) Integrate from $I = 0$ to $I = I$:

$$U = \int_0^I LI'\,dI' = L \cdot \frac{I'^2}{2}\Big|_0^I = \frac{1}{2}LI^2.$$

(e) **Units check:** $[L] = \mathrm{H}$ and $[I] = \mathrm{A}$:

$$[L][I]^2 = \mathrm{H} \cdot \mathrm{A}^2 = \frac{\mathrm{V} \cdot \mathrm{s}}{\mathrm{A}} \cdot \mathrm{A}^2 = \mathrm{V} \cdot \mathrm{A} \cdot \mathrm{s} = \mathrm{W} \cdot \mathrm{s} = \mathrm{J}.$$

(f) **Quick practice example:** For $L = 2.0\,\mathrm{H}$ and $I = 3.0\,\mathrm{A}$:

$$U = \frac{1}{2} \cdot 2.0\,\mathrm{H} \cdot (3.0\,\mathrm{A})^2 = 9.0\,\mathrm{J}.$$