Introduction to Capacitors and Inductors

August 9, 2025

1 Background

So far, we have considered two kinds of circuit elements, *batteries* & *resistors*. Any circuit element is defined by its V-I relationship, which quantifies how the voltage drop across that element depends on the current running through it. For a battery the V-I relationship is simply $V = \epsilon$, where ϵ is the electromotive force (constant voltage) of the battery. For a resistor, the V-I relationship is given by Ohm's law: $V = IR$.

Now, we're going to introduce two new circuit elements, that are very widely used: *capacitors* and *inductors*. You can see a schematic diagram of a capacitor and inductor in the figures below. So, what are they:

Capacitor: Physically, a capacitor consists of two large metal plates put closely together. A capacitor acts as a store of charge and is usually "charged up" by having positive charge loaded up on one of the plates and an equal and opposite negative charge on the other plate. That way, a capacitor is always charge neutral. *Don't worry if this does not make sense currently, it will become clearer below!*

Inductor: Physically, an inductor consists of a wire that is looped around itself many times, causing it to have *inductance*. We will talk more about what this means in our next module on electromagnetism. You can safely ignore understanding what inductance is at the moment.

So why introduce these circuit elements into our circuits? Well, it turns out they can be really useful for a lot of practical applications. We will not address that directly in this handout, but will learn to use and solve circuits with them. So for this, we are really only going to need to know about the relationship between current I and voltage V for these two circuit elements. This might feel a bit unsatisfying to you, as formulas will be pulled seemingly from nowhere, and it should! But I promise that at least for thecapacitor, motivation will be provided, and for the inductor, we will return to this in our next module.

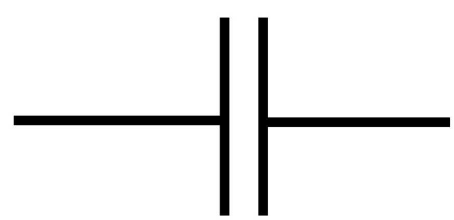

Figure 1: Capacitor (schematic)

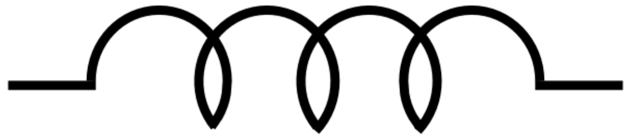

Figure 2: Inductor (schematic)

1.1 Capacitor

A capacitor acts as a store of charge. Technically, charge difference between its plates - the capacitor remains neutral. The higher the voltage across a capacitor, the larger the charge difference that can be maintained across the capacitor. The relationship between voltage and charge difference is the *capacitance* C, and is determined by the geometry and materials of the capacitor.

$$Q = CV$$

From this relationship, we can find the V-I relationship if we can just relate the charge Q on the capacitor to the current, I. It turns out that this is straightforward, by reasoning about how the charge arrives at a capacitor in a circuit. Charge arrives at one of the plates (and leaves the other) because there is a charge flowing into the capacitor form one of the sides (and out of the other). Hence, the current is nothing but the *rate of change of charge* on the capacitor. Thus:

$$I = \frac{dQ}{dT}$$

Which, together with the previous equation gives:

$$C\frac{dV}{dt} = I$$

This is our V-I relationship for a capacitor! *The significance of knowing this relationship will become apparent below, as we start solving circuit problems with capacitors.*

1.2 Inductor

The high-level view of how an inductor works is the following (don't worry if this doesn't make sense, we will be able to use the relationship nonetheless, and return to exactly how this works in our electromagnetism module):

> An inductor is a coil of wire looped around itself. As current flows through the wire, this generates a magnetic field (by Ampere's law). The rate of change of this magnetic field in turn generates an opposing voltage (Faraday's law), which opposes the current flow.

Hence, rate of change of current in an inductor causes an opposing voltage across an inductor, which seeks to resist the incoming current. The constant of proportionality between the rate of change of the current and the voltage is called the *inductance* L, and depends on the geometry and material of the inductor. So, our V-I relationship is:

$$V = -L\frac{dI}{dt}$$

Notice the minus sign which comes from the fact that the voltage opposes the incoming current. And notice the similarity with the capacitor, just with the time derivatives flipped! This is common in physics, another aspect of *duality* between two physical components.

1.3 Ordinary differential equations

The fact that the V-I relationship of the capacitor and inductor contain time derivatives mean that to solve Kirchoff's laws for the current and voltage, we will now have to deal with currents and voltages that *change* with time, and we will want to solve *differential equations* for I and V to solve the circuits.

So, as a quick primer on differential equations, in case you have forgotten (if you are seeing this for the first time, I would recommend consulting a high school or first-year university textbook for slightly more background, differential equations are fun and important!):

If we have a linear homogeneous (meaning that the right-hand side is 0) differential equation (which all of our circuit equations will be):

$$f(t) + \alpha \frac{df}{dt} = 0$$

We can rearrange (treating the differentials as fractions):

$$\beta dt = -\alpha \frac{df}{f}$$

we can then integrate:

$$\int_0^t dt' = -\alpha \int_{f(0)}^{f(t)} \frac{df}{f}$$

soling the integral:

$$t = -\alpha \ln \frac{f(t)}{f(0)}$$

$$f(t) = f(0)e^{-t/\alpha}$$

Voilà! We have solved for $f(t)$, and just need to put in the *initial condition* $f(0)$ to find our exact function.

In hindsight, we could have also solved this differently, by simply making the *ansatz* that $f(t) = Ae^{\lambda t}$. Plugging that into our first equation, we get:

$$Ae^{\lambda t} + \alpha\lambda Ae^{\lambda t} = 0$$

since this has to hold true for all t, we can divide out the exponentials and A, and get what is called the *characteristic* or *auxilliary* equation, which is justan algebraic equation for λ, which we can solve:

$$\lambda = -\frac{1}{\alpha}$$

which means that we recover our solution: $f(t) = Ae^{-t/\alpha}$

Importantly, this kind of ansatz **also works for second-order differential equations** (see question 6).

Now, what do we do in the case where we have an *inhomogeneous* differential equation, such as this one?

$$f(t) + \alpha \frac{df}{dt} = \beta$$

It turns out this is easy to solve once we've already solved the corresponding homogeneous equation[1]! Namely, it can be shown that the general solution to the inhomogenous solution can be written as just the sum of *any* particular solution to the inhomogeneous equation and the solution to the corresponding homogenous equation. So, we just find a constant solution to the homogenous equation (verify that this holds)

[1]The proof of this is available in most first-year university courses on ordinary differential equations. See for instance `https://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/ODE/2018/ODELectureNotes.pdf`, §2.4.3 for more insights.

$$f_{PI} = -\frac{\beta}{\alpha}$$

Now, our final inhomogeneous solution $f = f_{PI} + f_H$, where f_H is the solution to the homogeneous equation that we found above. So, our final solution to the inhomogeneous equation reads:

$$f = -\frac{\beta}{\alpha} + Ae^{-t/\alpha}$$

where A is a constant is to be determined by initial conditions.

Hence, we have a recipe for solving any linear differential equation:

1. Solve the corresponding honmogeneous equation, using your favorite technique (either rearrange treating the differentials as fractions, make an exponential ansatz, your favorite different technique!)
2. Find a particular solution to the inhomogeneous differential equation
3. Sum homogeneous and inhomogeneous solutions
4. Apply any initial conditions to determine remaining constants (there should be as many constants as the order of the equation, so 1 for first-order equations, 2 for second-order, etc).

This will be useful in *Try out the waters* and *Exploring the deep*, and elsewhere in life!

1.4 Summary

Here is the table of all the V-I relationship that we're not familiar. Now that we have these, we can attack many more circuits!

2 Questions

2.1 Derive the basics

1. **Equivalent capacitance of capacitors in series**: Consider the capacitors connected in series in figure 3. Let's find the equivalent capacitance of it!

   (a) If there is a charge Q on the leftmost place of capacitor C_1, what is the charge on the rightmost plate of capacitor C_2? *Hint: Consider that the amount of charge that has flowed into the connection from the left must equal the amount of charge that has flowed out of the circuit as a whole. Cahrge can only flow into the leftmost plate of C_1, and out of the rightmost plate of C_2.*

   (b) What is the charge on the rightmost plate of C_1 and C_2? *Hint: The total charge on a capacitor (right + left plate) must be neutral.*

   (c) What is the voltage across capacitor C_1? What is the voltage across C_2?

   (d) What is the total voltage across the capacitors in series? *Hint: Voltage adds in series.*

   (e) Now, we define the equivalent capacitance as $C_{eq} = \frac{Q}{V}$, where Q and V is the total charge that has flowed into the circuit connection, and V is the total voltage across the connection. Using this definition, and tehresults from above, find C_eq.

2. **Equivalent capacitance of capacitors in parallel**: Consider the capacitors connected in parallel in figure 4. Let's find the equivalent capacitance of it!

   (a) Call the total voltage drop across the circuit V. Knowing this, what is the charge Q_1 and Q_2 on each of teh capacitors? *Hint: The capacitors are connected in paralle. What does this mean for the voltages across each of the capacitors?*

   (b) What is the total charge Q that has flowed into the circuit, in terms of Q_1 and Q_2?

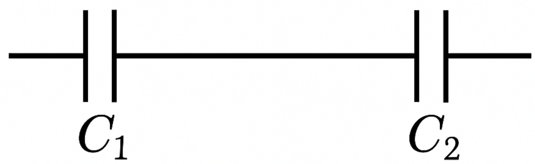

Figure 3: Capacitors in series

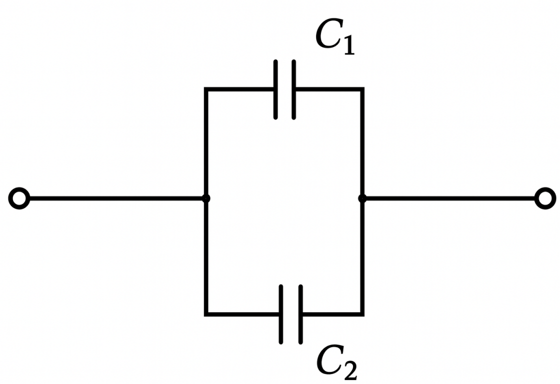

Figure 4: Circuits in parallel

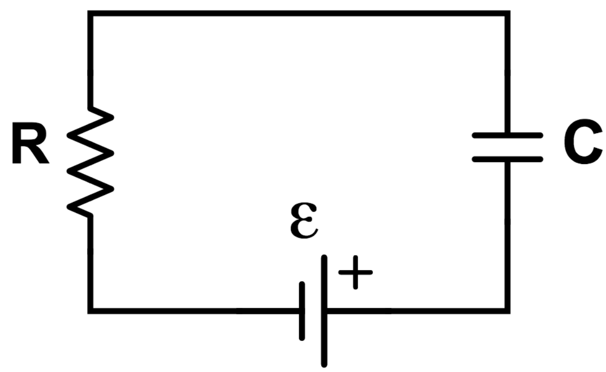

Figure 5: RC circuit

(c) Now, we define the equivalent capacitance as $C_{eq} = \frac{Q}{V}$, where Q and V is the total charge that has flowed into the circuit connection, and V is the total voltage across the connection. Using this definition, and tehresults from above, find C_eq.

3. Compare the results for C_{eq} found above for capacitors in series and in paralle. How do they compare to the expressions for equivalent resistance of *resistors* in series and in parallel?

2.2 Try out the waters

4. Consider a circuit consisting only of a battery, a resistor, and a capacitor, in series, as shown in figure 5.

   (a) Express the voltage $V(t)$ across the capacitor in terms of the capacitance C and the charge $Q(t)$ at a given time. *This is the V-I (or Q, rather) expressions above for the capacitor.*

   (b) Write down equation for the voltages across the circuit, using Kirchoff's voltage law. Express everything in terms of R, C, V, ϵ and Q.

   (c) Differentiate your entire expression with respect to time. Replace $\frac{dQ}{dt}$ with I.

   (d) Solve the first order differential equation for $I(t)$. From $I(t)$, find $V(t)$ using your expression from (b)

   (e) Sketch the solution of $V(t)$ and $I(t)$ vs t. What is V, I at $t = 0$ and $t \to \infty$?

   (f) What is the characteristic timescale τ at which $V(t)$ and $I(t)$ change over time, in terms of C amd R?

5. Consider a circuit consisting only of a battery, a resistor, and an inductor, in series, as shown in figure 6.

   (a) Express the voltage $V(t)$ across the inductor in terms of the inductance L and the time derivative of the current $\frac{dI(t)}{dt}$ at a given time. *This is the V-I expressions above for the inductor.*

   (b) Write down equation for the voltages across the circuit, using Kirchoff's voltage law. Express everything in terms of R, L, ϵ, I and $\frac{dI}{dt}$.

   (c) Solve the first order differential equation for $I(t)$. From $I(t)$, find $V(t)$ using your expression from (b)

   (d) Sketch the solution of $V(t)$ and $I(t)$ vs t. What is V, I at $t = 0$ and $t \to \infty$?

   (e) What is the characteristic timescale τ at which $V(t)$ and $I(t)$ change over time, in terms of L and R?

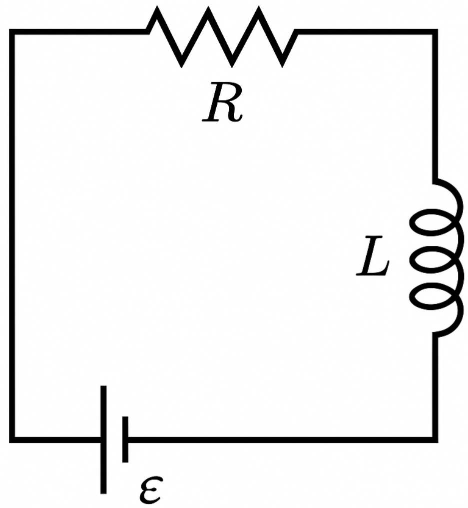

Figure 6: LR circuit

2.3 Exploring the deep

6. Consider a circuit consisting only of a capacitor and an inductor 7. At the start, the capacitor starts with a charge Q and there is no current in the circuit.

   (a) Express the voltage $V_L(t)$ across the inductor in terms of the inductance L and the time derivative of the current $\frac{dI(t)}{dt}$ at a given time.

   (b) Express the voltage $V_C(t)$ across the capacitor in terms of the inductance L and the charge $Q(t)$ at a given time.

   (c) Write down equation for the voltages across the circuit, using Kirchoff's voltage law. Express everything in terms of C, L, ϵ, Q and $\frac{dI}{dt}$.

   (d) Differentiate the expression from (c) with respect to time.

   (e) Solve the second-order differential equation for $I(t)$.

   (f) Sketch the solution of $V(t)$ and $I(t)$ vs t.

   (g) What is the characteristic timescale τ at which $V(t)$ and $I(t)$ change over time, in terms of L and C?

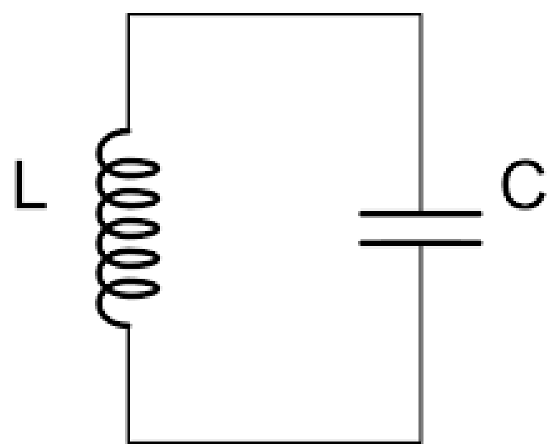

Figure 7: LC circuit